Path Algebras and BGP

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Think of a routing protocol as...

a distributed algorithm to solve an optimization problem

- Globally optimal paths
  - lots of theory (Dijkstra, Bellman-Ford, Semirings)
- Locally optimal paths
  - new territory
  - BGP belongs to this space
  - sufficient conditions known to guarantee a stable and unique solution

A Familiar Semiring

$(\mathbf{R}_+, +, \times)$

$2 + 3 = 3 + 2$	*Commutativity*
$(2 + 3) + 4 = 2 + (3 + 4)$	*Associativity*
$2 + 0 = 2$	*Identity*

$(2\cdot3)\cdot4 = 2\cdot(3\cdot4)$	*Associativity*
$2\cdot1 = 1\cdot2 = 2$	*Identity*
$2\cdot0 = 0\cdot2 = 0$	*Annihilator*

$2\cdot(3 + 4) = 2\cdot3 + 2\cdot4$	*Left Distributivity*
$(2 + 3)\cdot4 = 2\cdot4 + 3\cdot4$	*Right Distributivity*

$(\mathbf{S}, \oplus, \otimes)$

$a \oplus b = b \oplus a$	*Commutativity*
$(a \oplus b) \oplus c = a \oplus (b \oplus c)$	*Associativity*
$a \oplus 0 = a$	*Identity*

$(ab)c = a(bc)$	*Associativity*
$a1 = 1a = a$	*Identity*
$a0 = 0a = 0$	*Annihilator*

$a(b \oplus c) = ab \oplus ac$	*Left Distributivity*
$(b \oplus c)a = ba \oplus ca$	*Right Distributivity*

Semirings and Routing

S	⊕	⊗	Description
N ∪{+∞}	min	+	Shortest paths
N ∪{+∞}	max	min	Widest paths
[0, 1]	max	×	Most Reliable paths

Matrices can also form semirings!

Too good to be true ?!

Proving stability and uniqueness of paths, when:

- Requirements are more complicated
  - shortest widest path
  - BGP policies!
- Operator properties do not (need to) hold

  e.g lack of associativity or distributivity

Routing Algebra

- 3 sets:
  - Weights
  - Labels
  - Signatures, including *rejection*
- 1 operator: (label, signature) to signature
- 1 function: signature to weight
- a total order on weights

$$(W, \preceq, L, \Sigma, \phi, \oplus, f)$$

Example: business relationships

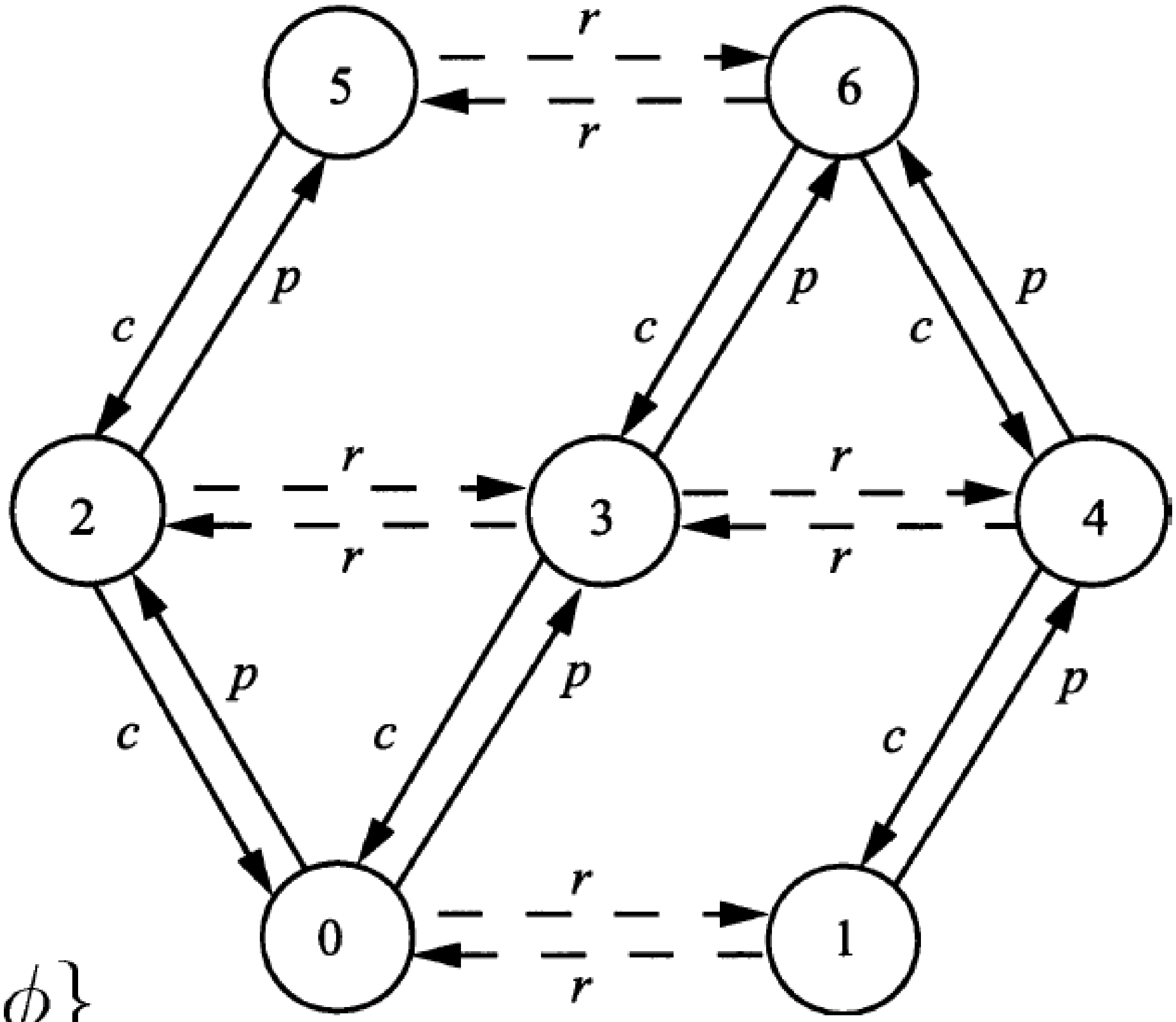
5 2 0 (c)

3 2 0 (r)

3 6 5 2 0 (p)

5 2 3 0

2 0 3



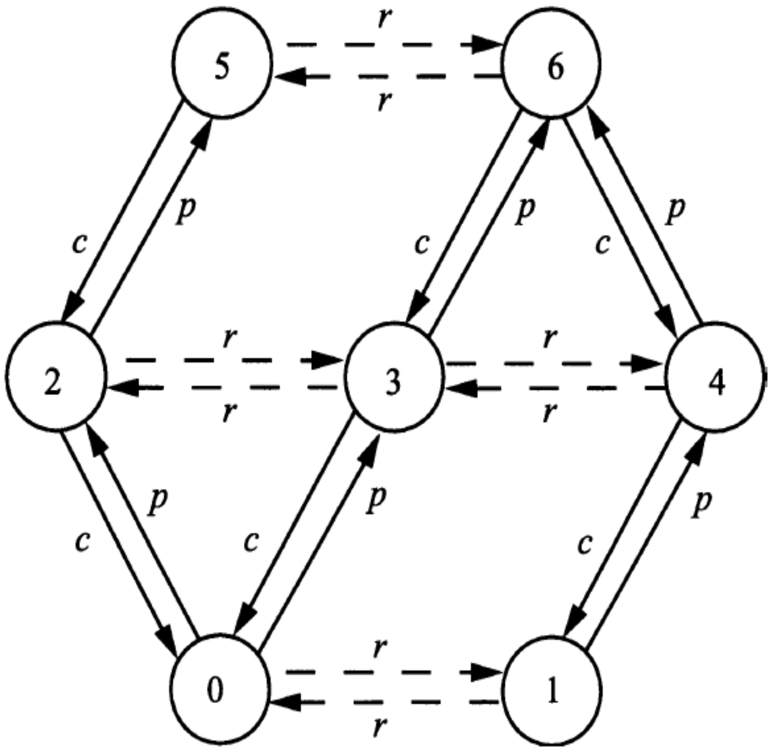
$$L = \{c, r, p\}$$
$$\Sigma = L \cup \{\epsilon, \phi\}$$

Formally

signature

label

$\oplus$	ϵ	c	r	p
c	c	c	ϕ	ϕ
r	r	r	ϕ	ϕ
p	p	p	p	p



$$f(c) = 1, \quad f(r) = f(p) = 2,$$

$$f(\epsilon) = 0, \quad f(\phi) = +\infty$$

the Gao-Rexford guideline/assumption:

$$f(c) \ = \ 1 \ < \ 2 \ = \ f(r) \ = \ f(p)$$

Convergence Properties

- Monotonicity (Strict) $l \in L \text{ and } \alpha \in \Sigma,$

$$f(\alpha) \preceq f(l \oplus \alpha)$$

- Isotonicity $l \in L \text{ and } \alpha, \beta \in \Sigma,$

$$f(\alpha) \preceq f(\beta) \Rightarrow f(l \oplus \alpha) \preceq f(l \oplus \beta)$$

- M: path vector protocols converge
  - Signaling information: (P, a)
- I: the paths onto they converge are optimal

Lexical Product

$A \otimes B$

$\oplus$	$(\sigma 1, \sigma 2)$
$(\lambda 1, \lambda 2)$	$(\lambda 1 \oplus \sigma 1, \lambda 2 \oplus \sigma 2)$

$(\phi, \_) = (\_, \phi) = \phi$

Preference is Lexical order

$$\langle \sigma_A,\ \sigma_B \rangle \preceq \langle \beta_A,\ \beta_B \rangle \stackrel{\text{def}}{=} \sigma_A \prec_A \beta_A \text{ or } (\sigma_A \sim_A \beta_A \text{ and } \sigma_B \preceq_B \beta_B)$$

Preservation properties

A	B	A ⊗ B
M	M	M
SM		SM
M	SM	SM

This suggests a design pattern for SM:

$$\underbrace{\underbrace{A1 \otimes A2 \otimes \dots Ai}_{\textbf{all M}} \otimes \underbrace{A(i+1)}_{\textbf{SM}} \otimes \underbrace{\dots \otimes An}_{\textbf{don't care}}}_{\textbf{SM}}$$

BGP could look like

$$\mathrm{EBGP}^{A} \quad \overset{\mathrm{def}}{=}$$

$$\begin{aligned} \mathrm{PROG}(\otimes(&\mathbf{locpref} : \mathrm{FLIP}(\mathrm{LP}(2^{32})),\\ &\mathbf{aspath} : \mathrm{SIMSEQ}(2^{16},\ 200),\\ &\mathbf{origin} : \mathrm{OP}(3),\\ &\mathbf{med} : \bot(\mathrm{min},\ \mathrm{LP}(2^{32})),\\ &\mathbf{community} : \bot(\mathrm{min},\ \mathrm{TAGS}(int)))) \end{aligned}$$

Since BGP paths are *locally* optimal

Distributivity is too strong a property

$$a \otimes (b \oplus c) \quad = \quad (a \otimes b) \oplus (a \otimes c)$$

Strictly Inflationary (remember SM) is sufficient for a stable, unique, locally optimal solution.

Important Properties

property	definition
D	$\forall a, b \in S,\ f \in F\ :\ f(a \oplus b) = f(a) \oplus f(b)$
INFL	$\forall a \in S,\ f \in F\ :\ a \leq f(a)$
S.INFL	$\forall a \in S,\ F \in F\ :\ a \neq \overline{0} \implies a < f(a)$
K	$\forall a, b \in S,\ f \in F\ :\ f(a) = f(b) \implies a = b$
$K_{\overline{0}}$	$\forall a, b \in S,\ f \in F\ :\ f(a) = f(b) \implies (a = b \vee f(a) = \overline{0})$
C	$\forall a, b \in S,\ f \in F\ :\ f(a) = f(b)$
$C_{\overline{0}}$	$\forall a, b \in S,\ f \in F\ :\ f(a) \neq f(b) \implies (f(a) = \overline{0} \vee f(b) = \overline{0})$

4! Inflationary Policy Functions for 3 strata

	0	1	2	D	K_∞	C_∞
a	0	1	2	$\star$	$\star$	
b	0	1	∞	$\star$	$\star$	
c	0	2	2	$\star$		
d	0	2	∞	$\star$	$\star$	
e	0	∞	2		$\star$	
f	0	∞	∞	$\star$	$\star$	$\star$
g	1	1	2	$\star$		
h	1	1	∞	$\star$		$\star$
i	1	2	2	$\star$		
j	1	2	∞	$\star$	$\star$	
k	1	∞	2		$\star$	
l	1	∞	∞	$\star$	$\star$	$\star$
m	2	1	2			
n	2	1	∞		$\star$	
o	2	2	2	$\star$		$\star$
p	2	2	∞	$\star$		$\star$
q	2	∞	2			$\star$
r	2	∞	∞	$\star$	$\star$	$\star$
s	∞	1	2		$\star$	
t	∞	1	∞		$\star$	$\star$
u	∞	2	2			$\star$
v	∞	2	∞		$\star$	$\star$
w	∞	∞	2		$\star$	$\star$
x	∞	∞	∞	$\star$	$\star$	$\star$

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